Exercise 2

A common inhabitant of human intestines is the bacterium *Escherichia coli*, named after the German pediatrician Theodor Escherich, who identified it in 1885. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 50 cells.

(a) Find the relative growth rate.

(b) Find an expression for the number of cells after t hours.

(c) Find the number of cells after 6 hours.

(d) Find the rate of growth after 6 hours.

(e) When will the population reach a million cells?

Solution

Part (a)

Assume that the relative growth rate is a constant k.

$$\frac{1}{P}\frac{dP}{dt} = k$$

Rewrite the left side using the chain rule.

$$\frac{d}{dt}\ln P = k$$

The function that you take a derivative of to get k is $kt + C$, where C is any constant.

$$\ln P = kt + C$$

Exponentiate both sides to get P.

$$\begin{aligned} P(t) &= e^{kt+C} \\ &= e^C e^{kt} \end{aligned}$$

Use a new constant P_0 for e^C.

$$P(t) = P_0 e^{kt} \tag{1}$$

Use the fact that the population doubles every 20 minutes, or $t = \frac{1}{3}$ hours.

$$P\left(\frac{1}{3}\right) = P_0 e^{k/3} = 2P_0$$

$$e^{k/3} = 2$$

$$\ln e^{k/3} = \ln 2$$

$$\frac{k}{3} = \ln 2$$

$$k = 3\ln 2 \approx 2.07944 \text{ hour}^{-1}$$

Part (b)

Equation (1) then becomes

$$P(t) = P_0 e^{(3\ln 2)t}. \tag{2}$$

Use the fact that the initial population is 50 to determine P_0.

$$P(0) = P_0 e^{(3\ln 2)0} = 50 \quad \rightarrow \quad P_0 = 50$$

Therefore,

$$P(t) = 50e^{(3\ln 2)t}.$$

Part (c)

After 6 hours the E. coli population is

$$P(6) = 50e^{(3\ln 2)6} \approx 13{,}107{,}200.$$

Part (d)

After 6 hours the rate of population growth is

$$\left.\frac{dP}{dt}\right|_{t=6} = kP(6) = (3\ln 2)(13{,}107{,}200) \approx 2.72557 \times 10^7 \ \frac{\text{cells}}{\text{hour}}.$$

Part (e)

To find when the population will reach 1,000,000, set $P(t) = 1{,}000{,}000$ and solve the equation for t.

$$P(t) = 1\,000\,000$$

$$50e^{(3\ln 2)t} = 1\,000\,000$$

$$e^{(3\ln 2)t} = 20\,000$$

$$\ln e^{(3\ln 2)t} = \ln 20\,000$$

$$(3\ln 2)t = \ln 20\,000$$

$$t = \frac{\ln 20\,000}{3\ln 2} \approx 4.76257 \text{ hours}$$